Announcements

1) Office hours unchanged this week.

Observations

1) Infinite differentiability

Any power series is infinitely differentiable on the open interval determined by its radius of convergence.

2) Complex numbers

Everything works the same, but more is true.

If f is a complex valued function of a complex variable, then if f is differentiable on an open subset of $\mathbb{C}$, containing 0 then $\exists$ complex numbers $a_0, a_1, a_2, \ldots$ such that

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z}$$

on that region.

Theorem: (differentiability)

Suppose $(f_n)_{n=1}^{\infty}$ is differentiable on $[a,b]$.

If $f_n' \to g$ uniformly on $[a,b]$ and $\exists\, x_0 \in [a,b]$ with $(f_n(x_0))_{n=1}^{\infty}$ convergent, then $\exists$ differentiable f on $[a,b]$, $f_n \to f$, $f' = g$.

<u>Proof</u>: 1^{st} issue - how to define f?

Choose $x \in [a,b]$, suppose $x > x_0$. Consider the sequence $(f_n(x))_{n=1}^{\infty}$.

We want to show the sequence is Cauchy, then define $f(x)$ as the limit.

By the mean value theorem, $\forall\ n, m \in \mathbb{N}$ $\exists\ c_{n,m} \in [x_0, x]$ satisfying

$$\frac{(f_n - f_m)(x) - (f_n - f_m)(x_0)}{x - x_0}$$

$$= (f_n - f_m)'(c_{n,m}).$$

Set $g_{n,m}(x) = (f_n - f_m)(x)$.

Then

$$g_{n,m}(x) - g_{n,m}(x_0)$$
$$= (x - x_0)\left(g'_{n,m}(c_{n,m})\right).$$

Now we have

$$g_{n,m}(x) = (x - x_0)(g'_{n,m})(c_{n,m})$$
$$+ g_{n,m}(x_0).$$

Since $(f_n')_{n=1}^{\infty}$ converges uniformly on $[a,b]$,

$(f_n')_{n=1}^{\infty}$ is uniformly Cauchy. Therefore, $\forall\ \varepsilon>0$

$\exists\ N_1 \in \mathbb{N}$ so that $\forall$

$n,m \geq N_1$,

$$|f_n'(y)-f_m'(y)| = |g_{n,m}'(y)| < \frac{\varepsilon}{2(x-x_0)}$$

$\forall\ y \in [a,b]$.

Then since

$(f_n(x_0))_{n=1}^{\infty}$ is convergent,

it is Cauchy, so $\exists\ N_2 \in \mathbb{N}$

such that $\forall\ n, m \geq N_2$,

$$|(f_n - f_m)(x_0)| = |g_{n,m}(x_0)| < \frac{\varepsilon}{2}.$$

Then

$$|(f_n - f_m)(x)|$$

$$= |g_{n,m}(x)|$$

$$\leq |g_{n,m}'(c_{n,m})|(x - x_0) + |g_{n,m}(x_0)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n,m \geq N = \max(N_1, N_2)$

Therefore, $(f_n(x))_{n=1}^{\infty}$

is Cauchy $\forall\ x > x_0$,

and the same argument

would work for $x < x_0$.

Hence, $(f_n(x))_{n=1}^{\infty}$

converges $\forall\ x \in [a, b]$;

we then define

$$\boxed{f(x) = \lim_{n \to \infty} f_n(x)}$$

Now to show $f' = g$!

Estimate

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right|$$

for a fixed $x \in [a,b]$

and show that as $h \to 0$,

the quantity approaches

zero.

$$\frac{f(x+h)-f(x)}{h} - g(x)$$

$$= \frac{f(x+h)-f(x)}{h} - f_n'(x)$$

$$+ f_n'(x) - g(x)$$

$$= \overbrace{\frac{f(x+h)-f(x)-f_n(x+h)+f_n(x)}{h}}^{\text{I}}$$

$$+ \overbrace{\frac{f_n(x+h)-f_n(x)}{h} - f_n'(x)}^{\text{II}}$$

$$+ \overbrace{f_n'(x) - g(x)}^{\text{III}}$$

II and III can be easily dispensed with.

We want to examine I.

Consider $g_{n,m}(x) = (f_n - f_m)(x)$.

Apply the mean value theorem to the interval with endpoints x and $x+h$ to obtain a $c = c(n,m,h)$ such that

$$\frac{g_{n,m}(x+h) - g_{n,m}(x)}{h}$$

$$= g_{n,m}'(c)$$

$$= (f_n' - f_m')(c).$$

Therefore, $\forall\ \varepsilon > 0,\ \exists\ N_1 \in \mathbb{N}$ such that $\forall\ n, m \geq N_1$,

$$|(f_n' - f_m')(y)| < \frac{\varepsilon}{3}$$

$\forall\ y \in [a, b]$.

This implies

$$\left| \frac{g_{n,m}(x+h) - g_{n,m}(x)}{h} \right| < \frac{\varepsilon}{3}$$

$\forall \; n,m \geq N_1$.

Take limit as $m \to \infty$.

Then

$$\lim_{m\to\infty} g_{n,m}(y) = \lim_{m\to\infty} \left(f_n(y) - f_m(y) \right)$$

$$= f_n(y) - f(y)$$

Then replacing f_m with f, we have

$$\frac{|(f-f_n)(x+h)-(f-f_n)(x)|}{|h|} < \frac{\varepsilon}{3} \quad \forall\ n \geq N_1 .$$

Now $\exists\ N_2 \in \mathbb{N}$

so that $\forall\ n \geq N_2$,

$$|f_n'(y) - g(y)| < \frac{\varepsilon}{3}$$

$\forall\ y \in [a,b]$.

Take $N = \max(N_1, N_2)$

Since $\dfrac{f_n(x+h) - f_n(x)}{h}$

$\to f_n'(x)$ as $h \to 0$,

$\exists$ $\delta > 0$ such that

$\forall$ h, $|h| < \delta$,

$$\left| \frac{f_N(x+h) - f_N(x)}{h} - f_N'(x) \right|$$

$$< \frac{\varepsilon}{3}$$

Then $\forall h$, $|h| < \delta$,

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right|$$

$$= \left| \frac{f(x+h) - f(x) - f_N(x+h) + f_N(x)}{h} \right|$$

$$+ \left| \frac{f_N(x+h) - f_N(x)}{h} - f_N'(x) \right|$$

$$+ \left| f_N'(x) - g(x) \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x),$$

which means $g(x) = f'(x)$. $\square$

Example: $\forall\ x,\ |x|<1,$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We also know

$$\frac{d}{dx}\left(\ln(|1-x|)\right) = \frac{1}{1-x}.$$

So then

$$\boxed{\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}$$

$\forall\ x,\quad |x|<1.$